On a Pólya functional for rhombi, isosceles triangles, and thinning convex sets.

M. van den Berg** , V. Ferone*, C. Nitsch*, C. Trombetti*

**School of Mathematics, University of Bristol
Fry Building, Woodland Road, Bristol BS8 1UG, UK
mamvdb@bristol.ac.uk

*Università degli Studi di Napoli Federico II
Via Cintia, Monte S. Angelo, I-80126 Napoli, Italy
vincenzo.ferone@unina.it
c.nitsch@unina.it
cristina.trombetti@unina.it

29 July 2019

Abstract

Let Ω be an open convex set in $\mathbb{R}^m$ with finite width, and with boundary $\partial\Omega$. Let v_Ω be the torsion function for Ω, i.e. the solution of $-\Delta v = 1, v|_{\partial\Omega} = 0$. An upper bound is obtained for the product of $\|v_\Omega\|_{L^\infty(\Omega)}\lambda(\Omega)$, where $\lambda(\Omega)$ is the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$. The upper bound is sharp in the limit of a thinning sequence of convex sets. For planar rhombi and isosceles triangles with area 1, it is shown that $\|v_\Omega\|_{L^1(\Omega)}\lambda(\Omega) \ge \frac{\pi^2}{24}$, and that this bound is sharp.

AMS 2000 *subject classifications.* 49J45, 49R05, 35P15, 47A75, 35J25.
Key words and phrases. torsion function, torsional rigidity, first Dirichlet eigenvalue

1 Introduction

Let Ω be an open set in Euclidean space $\mathbb{R}^m$, and with boundary $\partial\Omega$. We denote the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ by

$$\lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |D\varphi|^2}{\displaystyle\int_\Omega \varphi^2}.$$

It was shown in [3] and [1] that if

$$\lambda(\Omega) > 0,$$

then the torsion function, i.e. the unique weak solution of $v_\Omega : \Omega \mapsto \mathbb{R}^+$,

$$-\Delta v = 1,\ v|_{\partial\Omega} = 0,$$

satisfies

$$1 \le \lambda(\Omega) M(\Omega) \le c_m, \tag{1.1}$$

where
$$M(\Omega) = \|v_\Omega\|_{L^\infty(\Omega)}.$$
In [14] it was shown that
$$c_m \le \frac{1}{8}(m + (5(4+\log 2))^{1/2} m^{1/2} + 8)$$
The sharp constant in the right-hand side of (1.1) is not known. However, for an open ball $B \subset \mathbb{R}^2$, an open square S, and an equilateral triangle E,
$$\lambda(B)M(B) < \lambda(S)M(S) < \lambda(E)M(E). \tag{1.2}$$
The fact that $\lambda(B)M(B) < \lambda(E)M(E)$ was shown in [6]. The full inequality (1.2) follows from numerical evaluation of the series for the square, pp. 275-277 in [13].

In [2] it was shown that the left-hand side of (1.1) is sharp: for $\epsilon > 0, m \ge 2$, there exists an open bounded and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that
$$\lambda(\Omega_\epsilon)M(\Omega_\epsilon) < 1 + \epsilon.$$
For open, bounded convex sets in $\mathbb{R}^m$ it was shown in (3.12) of [8] that
$$\lambda(\Omega)M(\Omega) \ge \frac{\pi^2}{8}, \tag{1.3}$$
with equality in the limit of an infinite slab (the open set with finite width bounded by two parallel $(m-1)$-dimensional planes). The latter assertion has been made precise in [6] where it was shown that if
$$S_n = (-n, n)^{m-1} \times (0,1),\ n \ge 1, \tag{1.4}$$
then
$$\lambda(S_n)M(S_n) \le \frac{\pi^2}{8} + \frac{m-1}{8(n-\frac{2}{3})}. \tag{1.5}$$
For bounded planar convex sets with width $w(\Omega)$ and diameter $\mathrm{diam}(\Omega)$, it was shown in [2] that
$$\lambda(\Omega)M(\Omega) \le \frac{\pi^2}{8}\left(1 + 3^{2/3}7\left(\frac{w(\Omega)}{\mathrm{diam}(\Omega)}\right)^{2/3}\right). \tag{1.6}$$

Theorem 1.1 substantially improves upon (1.5) and (1.6). The former inequality holds only for a class of rectangular parallelepipeds, while the latter only holds for bounded planar convex sets.

It was shown in Theorem 1.5 in [4] that for an open, bounded, convex set Ω with finite width $w(\Omega)$, and boundary $\partial\Omega$ there exist two points $z_0 \in \partial\Omega$ and $z_1 \in \partial\Omega$ such that $|z_0 - z_1| = w(\Omega)$, and such that the two hyper-planes tangent to $\partial\Omega$ through z_0 and z_1 are parallel. Denote these two hyper-planes by H_0 and H_1 respectively. Denote the inradius of Ω and the centre of an inball by $r(\Omega)$ and $c(\Omega)$, respectively:
$$r(\Omega) = \sup_{x\in\Omega} \mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(c(\Omega), \partial\Omega).$$
We introduce Cartesian coordinates $(x', x_m) \in \mathbb{R}^m$ such that $x'(c(\Omega)) = 0, x_m(H_0) = 0, x_m(H_1) = w(\Omega)$. Let $H_\mu = \{x \in \mathbb{R}^m : x_m = \mu w(\Omega)\}$, $0 \le \mu \le 1$, and let
$$\Omega_c = \Omega \cap H_{x_m(c(\Omega))},$$
be the intersection of Ω and the hyper-plane through the centre of the inball and parallel to H_0. We denote the inradius of this $(m-1)$-dimensional set by $\rho(\Omega)$. If the points $z_0, z_1, c(\Omega)$ are not unique then we choose them such that $\rho(\Omega)$ is maximal. The measure of Ω is denoted by $|\Omega|$.

Theorem 1.1 *If Ω is an open, bounded, convex set in $\mathbb{R}^m, m \ge 2$, then*

$$\lambda(\Omega)M(\Omega) \le \frac{\pi^2}{8}\left(1 + d_m\left(\frac{w(\Omega)}{\rho(\Omega)}\right)^{2/3}\right), \tag{1.7}$$

where

$$d_m = 7(m+1)^{4/3}\pi^{-2}j^2_{(m-3)/2}, \tag{1.8}$$

and where j_ν is the first positive zero of the Bessel function J_ν.

We note that the role of $\mathrm{diam}(\Omega)$ in (1.6) has, in Theorem 1.1, been replaced by the inradius of Ω_c. It is easily seen that these two quantities are comparable in the planar case. However, in the higher-dimensional case one requires that, in order to have the upper bound close to $\frac{\pi^2}{8}$, all length scales of the projection are large compared with the width. This is certainly satisfied if $\rho(\Omega)$ is large compared with $w(\Omega)$.

The remaining results of this paper are for the Pólya functional for isosceles triangles and rhombi. Recall that the torsional rigidity (or torsion) $T(\Omega)$ of an open set Ω is defined by

$$T(\Omega) = \|v_\Omega\|_{L^1(\Omega)} = \int_\Omega v_\Omega. \tag{1.9}$$

In Pólya and Szegö [9], it was shown that for sets Ω with finite measure $|\Omega|$,

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \le 1. \tag{1.10}$$

The left-hand side of (1.10) is the Pólya functional for Ω. It was subsequently shown in [4] that the constant 1 in the right-hand side above is sharp: for $\epsilon > 0$, there exists an open, bounded, and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that $T(\Omega_\epsilon)\lambda(\Omega_\epsilon)|\Omega_\epsilon|^{-1} \ge 1 - \epsilon$.

The left-hand side of (1.10) is invariant under the homothety transformation $t \mapsto t\Omega$. This implies for example that in Theorems 1.2-**??** below we do not have to specify the actual lengths of the edges of the rhombi and triangles. In the proofs of these theorems we fix the various lengths as a matter of convenience.

It was shown in Theorem 1.5 of [4] that for a thinning (collapsing) sequence (Ω_n) of bounded convex sets

$$\limsup_{n\to\infty} \frac{T(\Omega_n)\lambda(\Omega_n)}{|\Omega_n|} \le \frac{\pi^2}{12}. \tag{1.11}$$

This supports the conjecture that for bounded, convex sets the sharp constant in the right-hand side of (1.10) is $\pi^2/12$.

It was shown in Theorem 1.4 in [4] that for bounded convex sets in $\mathbb{R}^m, m \ge 3$,

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \ge \frac{\pi^2}{4m^{m+2}(m+2)}, \tag{1.12}$$

and that for planar, bounded, convex sets,

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \ge \frac{\pi^2}{48}. \tag{1.13}$$

In Theorems 1.2 and 1.3 we show that for isosceles triangles and rhombi the constant in the right-hand side of (1.13) can be improved to $\pi^2/24$, and that this constant is sharp.

Theorem 1.2 *If $\triangle_\beta$ is an isosceles triangle with angles $\beta, \beta, \pi - 2\beta$, then*

$$\frac{\pi^2}{24} \le \frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} \le \frac{\pi^2}{24}\big(1 + 8\big(\tan\beta\big)^{2/3}\big). \tag{1.14}$$

Theorem 1.3 *If $\lozenge_\beta$ is a rhombus with angles $\beta, \pi-\beta, \beta, \pi-\beta$, then*

$$\frac{\pi^2}{24} \le \frac{T(\lozenge_\beta)\lambda(\lozenge_\beta)}{|\lozenge_\beta|} \le \frac{\pi^2}{24}\big(1+8\big(\tan(\beta/2)\big)^{2/3}\big). \tag{1.15}$$

This paper is organised as follows. In Section 2 we prove Theorem 1.1. The proofs of the upper bounds in Theorems 1.2 and 1.3 are deferred to Section 3. The proof of the lower bound in Theorem 1.3 is deferred to Section 4. The proof of lower bound in Theorem 1.2 consists of two parts. In Section 5 part 1 we show that the lower bound in (1.14) holds for all $\beta \in (0,\pi/3] \cup [\beta_0, \pi/2)$, where

$$\beta_0 = \frac{\pi}{2} - \frac{33}{200}. \tag{1.16}$$

In Section 5 part 2 we use interval arithmetic to verify that the lower bound in (1.14) also holds for $\beta \in (\pi/3, \beta_0)$.

2 Proof of Theorem 1.1

Proof of Theorem 1.1. We first observe that by domain monotonicity of the torsion function, v_Ω is bounded by the torsion function for the (connected) set bounded by H_0 and H_1. Hence

$$v_\Omega(x) \le \frac{1}{2}x_m(w(\Omega)-x_m) \le \frac{w(\Omega)^2}{8},\ (x', x_m) \in \Omega.$$

It suffices to obtain an upper bound for $\lambda(\Omega)$. By convexity we have that the convex hull of z_0, z_1, Ω_c is contained in Ω. This convex hull in turn contains a cylinder with height $z \in [0, w(\Omega)]$, and base $\left(1-\frac{z}{w(\Omega)}\right)\Omega_c$. Denote the first $(m-1)$-dimensional Dirichlet eigenvalue of Ω_c by λ_c. Then, by separation of variables, we have

$$\lambda(\Omega) \le \frac{\pi^2}{z^2} + \left(1-\frac{z}{w(\Omega)}\right)^{-2}\lambda_c. \tag{2.1}$$

The right-hand side of (2.1) is minimised for

$$\frac{1}{z} = \frac{1}{w(\Omega)} + \left(\frac{\lambda_c}{\pi^2 w(\Omega)}\right)^{1/3}.$$

This gives that

$$\lambda(\Omega) \le \frac{\pi^2}{w(\Omega)^2}\left(1+3\left(\frac{\lambda_c w(\Omega)^2}{\pi^2}\right)^{1/3} + 3\left(\frac{\lambda_c w(\Omega)^2}{\pi^2}\right)^{2/3} + \frac{\lambda_c w(\Omega)^2}{\pi^2}\right). \tag{2.2}$$

The inball intersects Ω_c in a $(m-1)$-dimensional disc with radius $r(\Omega)$ which is, by a generalisation of Blaschke's theorem (see p.215 in [15], and p.79 in [7]), bounded from below by $w(\Omega)/(m+1)$. Hence

$$\lambda_c \le (m+1)^2 j_{(m-3)/2}^2 w(\Omega)^{-2}. \tag{2.3}$$

By (2.2) and (2.3) we obtain

$$\begin{aligned}\lambda(\Omega) &\le \frac{\pi^2}{w(\Omega)^2}\left(1+\left(\frac{\lambda_c w(\Omega)^2}{\pi^2}\right)^{1/3}\left(3+3\left(\frac{(m+1)^2 j_{(m-3)/2}^2}{\pi^2}\right)^{1/3} + \left(\frac{(m+1)^2 j_{(m-3)/2}^2}{\pi^2}\right)^{2/3}\right)\right)\\ &\le \frac{\pi^2}{w(\Omega)^2}\left(1+7\left(\frac{(m+1)^2 j_{(m-3)/2}^2}{\pi^2}\right)^{2/3}\left(\frac{\lambda_c w(\Omega)^2}{\pi^2}\right)^{1/3}\right).\end{aligned} \tag{2.4}$$

where we have used that $(m+1)^2 j_{(m-3)/2}^2 \ge \pi^2$. The latter inequality follows for $m = 2,3,4$ by inspection since $j_{-1/2} = j_{1/2} = \pi$, $j_0 = 2.4048...$ For $m \ge 5$ we have by [10] that $j_{(m-3)/2} \ge (m-3)/2 \ge 1$. Since the $(m-1)$-dimensional set Ω_c contains a disc of radius $\rho(\Omega)$ we have

$$\lambda_c \le j_{(m-3)/2}^2 \rho(\Omega)^{-2}, \tag{2.5}$$

and (1.7), (1.8) follows by (2.4) and (2.5). $\square$

3 Proofs of the upper bounds in Theorems 1.2 and 1.3

Proof of the upper bound in Theorem 1.2. Let $\triangle_\beta$ be an isosceles triangle with a base of length 2 and height of length d, and angles β, β, and $\pi - 2\beta$ respectively. First consider the case $\tan\beta \ge \frac{1}{8}$. Then the right-hand side of (1.14) is greater or equal than 1, and the inequality holds trivially by (1.10). Next consider the case $0 < \tan\beta \le \frac{1}{8}$. Then $d \le \frac{1}{8}$. We denote the infinite sector with opening angle β in polar coordinates (r, ϕ) by

$$\Omega_\beta = \{(r, \phi) : r > 0, -\beta/2 < \phi < \beta/2\}.$$

It is straightforward to verify that the torsion function for Ω_β is given by

$$v_{\Omega_\beta}(r, \phi) = \frac{r^2}{4}\left(\frac{\cos(2\phi)}{\cos\beta} - 1\right),\ r > 0, -\beta/2 < \phi < \beta/2.$$

Let

$$R = \left(1 + d^2\right)^{1/2}.$$

We can cover $\triangle_\beta$ with two sectors of opening angles β and radii R each. By monotonicity and positivity of the torsion function we have

$$\begin{aligned} T(\triangle_\beta) &= \int_{\triangle_\beta} v_{\triangle_\beta} \\ &\le 2\int_0^R dr\, r \int_{-\beta/2}^{\beta/2} d\phi\, v_{\Omega_\beta}(r, \phi) \\ &= \frac{1}{8}\left(1 + d^2\right)^2\left(\tan\beta - \beta\right) \\ &= \frac{1}{8}\left(1 + d^2\right)^2(d - \arctan d) \\ &\le \frac{d^3}{24}\left(1 + d^2\right)^2, \end{aligned} \tag{3.1}$$

where we have used that $d - \arctan d \le d^3/3$. By adapting formula (31) in the proof of Theorem 2 in [2] to the geometry of $\triangle_\beta$ we find that

$$\lambda(\triangle_\beta) \le \frac{\pi^2}{d^2}\left(1 + 7\left(\frac{d}{2}\right)^{2/3}\right). \tag{3.2}$$

By (3.1), (3.2), and $|\triangle_\beta| = d$, we obtain

$$\begin{aligned} \frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} &\le \frac{\pi^2}{24}\left(1 + d^2\right)^2\left(1 + 7\left(\frac{d}{2}\right)^{2/3}\right) \\ &= \frac{\pi^2}{24}\left(1 + \left(2d^{4/3} + d^{10/3} + 7\cdot 2^{-2/3}(1 + 2d^2 + d^4)\right)d^{2/3}\right) \\ &\le \frac{\pi^2}{24}\left(1 + 8d^{2/3}\right) \\ &= \frac{\pi^2}{24}\left(1 + 8\left(\tan\beta\right)^{2/3}\right), \end{aligned} \tag{3.3}$$

where we have used that $2d^{4/3} + d^{10/3} + 7\cdot 2^{-2/3}(1 + 2d^2 + d^4) < 8$ for $d \le \frac{1}{8}$. □

Proof of the upper bound in Theorem 1.3. Let $\lozenge_\beta$ be a rhombus with angles $\beta, \pi - \beta, \beta, \pi - \beta$, and diagonals of length 2 and d respectively. First consider the case $d \ge \frac{1}{4}$. Then $\tan(\beta/2) \ge \frac{1}{8}$, and the right-hand side of (1.15) is greater or equal than 1. The inequality holds trivially by (1.10). Next consider the case $d \le \frac{1}{4}$. This rhombus is covered by two sectors of opening angle $\beta = 2\arctan(d/2)$, and radius $R = (1+(d/2)^2)^{1/2}$. By the calculations in the proof of Theorem 1.2 we find that

$$\begin{aligned}
T(\Diamond_\beta) &\le \frac{1}{8}R^4(\tan\beta - \beta)\\
&= \frac{1}{8}\Big(1+\frac{d^2}{4}\Big)^2\Big(\frac{d}{1-\frac{d^2}{4}} - 2\arctan\big(\frac{d}{2}\big)\Big)\\
&\le \frac{1}{8}\Big(1+\frac{d^2}{4}\Big)^2\Big(\frac{d}{1-\frac{d^2}{4}} - d + \frac{d^3}{12}\Big)\\
&= \frac{d^3}{24}\Big(1+\frac{d^2}{4}\Big)^2\frac{1-\frac{d^2}{16}}{1-\frac{d^2}{4}}\\
&\le \frac{d^3}{24}\Big(1+\frac{d^2}{4}\Big)^2\Big(1+\frac{4d^2}{21}\Big),
\end{aligned}$$

where we have used that $(1-d^2/16)(1-d^2/4)^{-1} \le 1 + 4d^2/21, 0 < d \le \frac{1}{4}$. By adapting formula (31) in the proof of Theorem 2 in [2] to the geometry of $\Diamond_\beta$ we find that

$$\lambda(\Diamond_\beta) \le \frac{\pi^2}{d^2}\bigg(1+7\Big(\frac{d}{2}\Big)^{2/3}\bigg).$$

This, together with $|\Diamond_\beta| = d$ gives that,

$$\begin{aligned}
\frac{T(\Diamond_\beta)\lambda(\Diamond_\beta)}{|\Diamond_\beta|} &\le \frac{\pi^2}{24}\Big(1+\frac{d^2}{4}\Big)^2\Big(1+\frac{4d^2}{21}\Big)\bigg(1+7\Big(\frac{d}{2}\Big)^{2/3}\bigg)\\
&= \frac{\pi^2}{24}\Bigg(1+\bigg(7\Big(1+\frac{d^2}{4}\Big)^2\Big(1+\frac{4d^2}{21}\Big) + 2^{2/3}\Big(\frac{29d^{4/3}}{42}+\frac{53d^{10/3}}{336}+\frac{d^{16/3}}{84}\Big)\bigg)\Big(\frac{d}{2}\Big)^{2/3}\Bigg)\\
&\le \frac{\pi^2}{24}\bigg(1+8\Big(\frac{d}{2}\Big)^{2/3}\bigg)\\
&= \frac{\pi^2}{24}\Big(1+8\big(\tan(\beta/2)\big)^{2/3}\Big),
\end{aligned} \tag{3.4}$$

where we have bounded the term in front of the $(d/2)^{2/3}$ in the second line of the right-hand side of (3.4) using $d \le \frac{1}{4}$. □

4 Proof of the lower bound in Theorem 1.3

Let $\Diamond_\beta$ be a rhombus such that major and minor diagonals have lengths 2 and d, respectively (see Figure 1). We want to estimate the torsion and to this aim we use a test function

$$v(x,y) = \begin{cases} \dfrac{d^2x^2}{4} - y^2, & 0 \le x \le 1,\\[2ex] \dfrac{d^2(2-x)^2}{4} - y^2, & 1 \le x \le 2. \end{cases}$$

In view of the variational definition of the torsion (see for example Chapter V in [9]) we have, by a straightforward calculation,

$$\frac{1}{T(\Diamond_\beta)} \le \frac{\displaystyle\int_{\Diamond_\beta}|Dv|^2}{\bigg(\displaystyle\int_{\Diamond_\beta} v\bigg)^2} = \frac{24+18d^2}{d^3}.$$

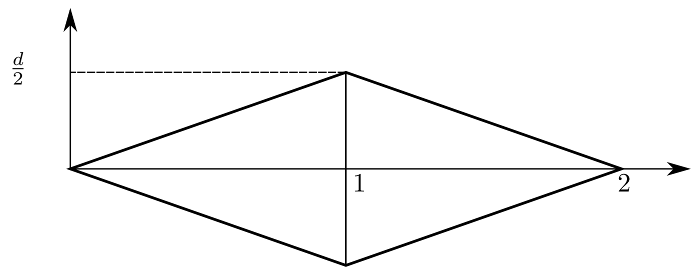

Figure 1: The rhombus of diagonals 2 and d. To estimate the torsion we construct a test function v which is symmetric with respect to the minor diagonal.

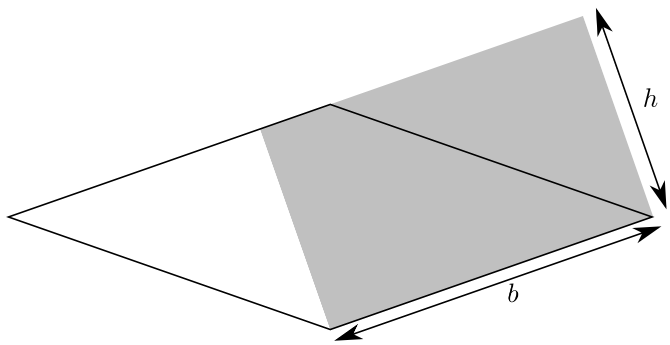

Figure 2: The rectangle shaded in grey is obtained by Steiner symmetrization. The Dirichlet Laplacian eigenvalue of the rectangle provides an estimate from below for the one on the rhombus.

On the other hand we can estimate from below the first Dirichlet Laplacian eigenvalue of any rhombus by means of the Dirichlet Laplacian eigenvalue of a rectangle obtained by Steiner symmetrising the rhombus along a direction parallel to one of the sides (see Figure 2). This is the very same idea that was used by Pólya to deduce that, for any given quadrilateral, there exists a rectangle of same area having a smaller first Dirichlet Laplacian eigenvalue (see for instance [5, Theorem 3.3.3]).
We denote by b and h the base and the height of the rectangle, respectively. Since the base b coincides with the side of the rhombus, $b^2 = 1 + \frac{d^2}{4}$, and $h = \frac{d}{\sqrt{1+\frac{d^2}{4}}}$.

We have,

$$\lambda(\Diamond_\beta) \geq \pi^2 \left(\frac{1}{b^2} + \frac{1}{h^2} \right) = \pi^2 \frac{16 + 24d^2 + d^4}{d^2(16 + 4d^2)}.$$

Observing that the area of the rhombus is equal to d, we have

$$\frac{\lambda(\Diamond_\beta) T(\Diamond_\beta)}{|\Diamond_\beta|} \geq \frac{\pi^2}{24} \frac{16 + 24d^2 + d^4}{(1 + \frac{3}{4}d^2)(16 + 4d^2)} \geq \frac{\pi^2}{24}, \qquad 0 \leq d \leq 2. \tag{4.1}$$

$\square$

5 Proof of the lower bound in Theorem 1.2

5.1 Proof for the case $\beta \in (0, \pi/3] \cup [\beta_0, \pi/2)$

Let $\triangle_\beta$ be an isosceles triangle with angles $\beta, \beta, \alpha = \pi - 2\beta$. We first consider the case $\frac{\pi}{3} \leq \alpha < \pi$. We denote the height by h, and we fix the length of the basis equal to 2. See Figure 3.

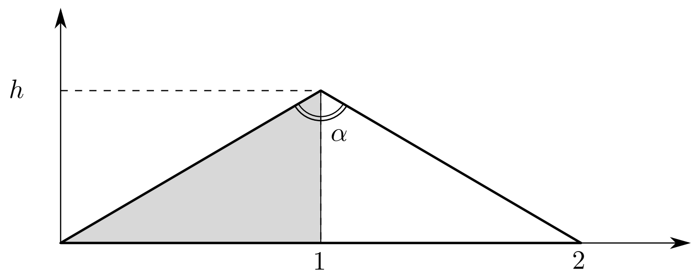

Figure 3: Isosceles triangle of basis 2, vertex angle α and height h. A test function to estimate the torsion is constructed on the shaded part and symmetrically reflected along the height.

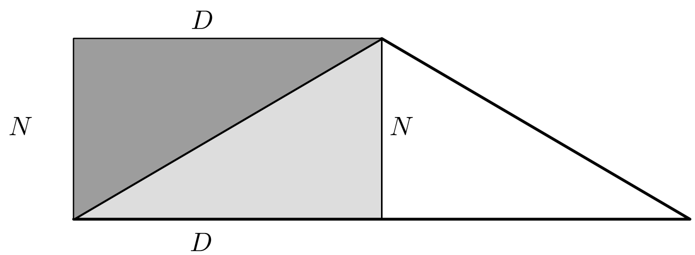

Figure 4: In the Figure the letter D and N corresponds to Dirichlet boundary conditions and Neumann boundary conditions respectively. On such a rectangle a test function is provided by reflecting (anti-symmetrically) the eigenfunction of the triangle, from the light grey part to the dark grey part.

We use the function

$$u(x,y) = \begin{cases} \dfrac{h^2x^2}{4} - \left(y - \dfrac{hx}{2}\right)^2, & 0 \le x \le 1, \\ \\ \dfrac{h^2(2-x)^2}{4} - \left(y - \dfrac{h(2-x)}{2}\right)^2, & 1 \le x \le 2, \end{cases}$$

as a test function for the torsion of $\triangle_\beta$. We find that

$$\frac{2}{T(\triangle_\beta)} \le \frac{48(1+h^2)}{h^3}. \tag{5.1}$$

Hence

$$\frac{T(\triangle_\beta)}{|\triangle_\beta|} \ge \frac{1}{24}\left(1 + \frac{1}{h^2}\right)^{-1}. \tag{5.2}$$

We wish to estimate $\lambda(\triangle_\beta)$ from below. To this aim we consider the first Dirichlet eigenfunction of $\triangle_\beta$ restricted to $x \in [0,1]$ and we reflect it, anti-symmetrically, with respect to the line $y = hx$ (see Figure 4). This new function is a test function defined on the rectangle of sides $1, h$ (shaded in grey in Figure 4) orthogonal to the first eigenfunction of the Laplacian with the mixed boundary conditions described in Figure 4.

For $\dfrac{\pi}{3} \le \alpha \le \pi$ we find that $h \le \sqrt{3}$, and

$$\lambda(\triangle_\beta) \ge \min\left\{\pi^2\left(1 + \frac{1}{h^2}\right), \frac{4\pi^2}{h^2}\right\} = \pi^2\left(1 + \frac{1}{h^2}\right). \tag{5.3}$$

Combining (5.2) and (5.3) we obtain

$$\frac{T(\triangle_\beta)\lambda(\triangle_\beta))}{|\triangle_\beta|} \ge \frac{\pi^2}{24},\ 0 < \beta \le \frac{\pi}{3}.$$

Next we consider the case $0 < \alpha \leq \dfrac{\pi}{3}$ or $\pi/3 \leq \beta < \pi/2$. We have

$$|\triangle_\beta| = 1/\tan(\alpha/2). \tag{5.4}$$

Let

$$S(\rho,\alpha) = \{(r,\phi) : 0 < r < \rho,\ -\alpha/2 < \phi < \alpha/2\}$$

be the circular sector with radius ρ and opening angle α. Siudeja's Theorem 1.3 in [11] asserts that for $0 < \beta \leq \pi/3$, $\lambda(\triangle_{\pi/2-\alpha/2}) \geq \lambda(S(\rho,\alpha))$, where $d\rho$ is such that $|\triangle_\beta| = |S(\rho,\alpha)|$. It follows that

$$\rho^2 = 2/(\alpha\tan(\alpha/2)). \tag{5.5}$$

Hence

$$\lambda(\triangle_\beta) \geq 2^{-1}\alpha\tan(\alpha/2) j^2_{\pi/\alpha}.$$

where we have used that the first Dirichlet eigenvalue of a circular sector of opening angle β and radius ρ equals $j^2_{\pi/\beta}\rho^{-2}$. See [9]. Moreover by (1.2) and (4.3) for $k = 1$ and $\nu = \pi/\alpha$ in [10] we have

$$j^2_{\pi/\alpha} > \left(\frac{\pi}{\alpha} - \frac{a_1}{2^{1/3}}\left(\frac{\pi}{\alpha}\right)^{1/3}\right)^2, \quad -a_1 \geq \left(\frac{9\pi}{8}\right)^{2/3},$$

where a_1 is the first negative zero of the Airy function. It follows that

$$j^2_{\pi/\alpha} \geq \frac{\pi^2}{\alpha^2}\left(1 + C\left(\frac{\alpha}{\pi}\right)^{2/3}\right)^2 \geq \frac{\pi^2}{\alpha^2}\big(1 + C_1\alpha^{2/3}\big), \tag{5.6}$$

where

$$C = (9\pi/8)^{2/3}2^{-1/3},\ C_1 = (9/4)^{2/3}. \tag{5.7}$$

The torsion function for $S(\rho,\alpha)$, $\alpha < \pi/2$, is given by (p.279 in [13]),

$$\begin{aligned} v_{S(\rho,\alpha)}(r,\phi) =& \frac{r^2}{4}\left(\frac{\cos(2\phi)}{\cos\alpha} - 1\right) \\ &+ \frac{4\rho^2\alpha^2}{\pi^3}\sum_{n=1,3,5,\ldots}(-1)^{(n+1)/2}\left(\frac{r}{\rho}\right)^{n\pi/\alpha}\cos\left(\frac{n\pi\phi}{\alpha}\right)n^{-1}\left(n + \frac{2\alpha}{\pi}\right)^{-1}\left(n - \frac{2\alpha}{\pi}\right)^{-1}. \end{aligned}$$

By monotonicity of the torsion we obtain

$$\begin{aligned} T(\triangle_\beta) &\geq T(S(\rho,\alpha)) \\ &= \int_{(0,d)} r\,dr \int_{(-\alpha/2,\alpha/2)} d\phi\, v_{S(d,\alpha)}(r,\phi) \\ &= \frac{d^4}{16}\left(\tan\alpha - \alpha - \frac{128\alpha^4}{\pi^5}\sum_{n=1,3,..} n^{-2}\left(n + \frac{2\alpha}{\pi}\right)^{-2}\left(n - \frac{2\alpha}{\pi}\right)^{-1}\right), \end{aligned} \tag{5.8}$$

We have that for $0 < \alpha \leq \pi/3$, $(n + 2\alpha/\pi)^2(n - 2\alpha/\pi) \geq \frac{25}{27}n^3$, $n \in \mathbb{N}$. This gives that

$$\begin{aligned} T(\triangle_\beta) &\geq \frac{\rho^4}{16}\left(\tan\alpha - \alpha - \frac{2^2 3^3 31\zeta(5)\alpha^4 d^4}{25\pi^5}\right) \\ &\geq \frac{\alpha^3\rho^4}{48}\big(1 - C_2\alpha\big), \end{aligned} \tag{5.9}$$

where

$$C_2 = \frac{2^2 3^4 31\zeta(5)}{5^2\pi^5}.$$

By (5.6), (5.8), (5.9), and (5.4) we obtain

$$\frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} \geq \frac{\pi^2}{24}\big(1 - C_2\alpha\big)\big(1 + C_1\alpha^{2/3}\big). \tag{5.10}$$

The right-hand side of (5.10) is greater or equal than $\frac{\pi^2}{24}$ for

$$C_1 \geq C_1 C_2 \alpha + C_2 \alpha^{1/3}. \tag{5.11}$$

Inequality (5.11) holds for all $\alpha \leq 33/100$.

5.2 Computer validation for the case $\beta \in (\pi/3, \beta_0)$ via interval arithmetic.

We consider a triangle $\triangle^\alpha$ of height 1 and opening angle α, where $\alpha = \pi - 2\beta$. Let

$$F(\alpha) = \frac{24}{\pi^2}\frac{\lambda(\triangle^\alpha)T(\triangle^\alpha)}{|\triangle^\alpha|}.$$

We wish to show that $F(\alpha) > 1.01$ in the range $0.33 \leq \alpha \leq \pi/3$.

We present here a computer assisted proof of the result using Interval Arithmetic.

We once more use Siudeja's lower bound, comparing with the sector having the same opening angle and the same area (Theorem 1.3 of [11]), and get

$$\lambda(\triangle^\alpha) \geq \tilde{\lambda}(\alpha) = \cos^2\left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{\sin\alpha}\right)\left(\frac{\pi}{\alpha} + C\left(\frac{\pi}{\alpha}\right)^{\frac{1}{3}}\right)^2, \tag{5.12}$$

where C is given by (5.7).

The area is given by

$$|\triangle^\alpha| = \tan\left(\frac{\alpha}{2}\right), \tag{5.13}$$

The monotonicity of T with respect to inclusion allows us to estimate from below using the torsion of a tangent sector with same opening angle α. We use (5.8) and find that

$$T(\triangle^\alpha) \geq \frac{1}{16}(\tan\alpha - \alpha) - \frac{8}{\pi^5}\alpha^4 \sum_{n=1,3,5,\dots} n^{-2}\left(n + \frac{2\alpha}{\pi}\right)^{-2}\left(n - \frac{2\alpha}{\pi}\right)^{-1}.$$

In order to perform a numerical evaluation we truncate the series in the following way

$$\sum_{n=1,3,5,\dots} n^{-2}\left(n + \frac{2\alpha}{\pi}\right)^{-2}\left(n - \frac{2\alpha}{\pi}\right)^{-1} = \sum_{n=0}^{\infty}(2n+1)^{-2}\left(2n+1+\frac{2\alpha}{\pi}\right)^{-2}\left(2n+1-\frac{2\alpha}{\pi}\right)^{-1}$$

$$\leq \sum_{n=0}^{N}(2n+1)^{-2}\left(2n+1+\frac{2\alpha}{\pi}\right)^{-2}\left(2n+1-\frac{2\alpha}{\pi}\right)^{-1} + \frac{1}{2^5}\sum_{n=N+1}^{\infty}\frac{1}{n^5}$$

$$\leq \sum_{n=0}^{N}(2n+1)^{-2}\left(2n+1+\frac{2\alpha}{\pi}\right)^{-2}\left(2n+1-\frac{2\alpha}{\pi}\right)^{-1} + \frac{1}{2^7 N^4}.$$

It follows that

$$\begin{aligned} T(\triangle^\alpha) &\geq \tilde{T}(\triangle^\alpha) \\ &= \frac{1}{16}(\tan\alpha - \alpha) - \frac{8}{\pi^5}\alpha^4\left(\sum_{n=0}^{10}(2n+1)^{-2}\left(2n+1+\frac{2\alpha}{\pi}\right)^{-2}\left(2n+1-\frac{2\alpha}{\pi}\right)^{-1} + \frac{1}{2^7\cdot 10^4}\right). \end{aligned} \tag{5.14}$$

Therefore

$$F(\alpha) \geq G(\alpha) = \frac{24}{\pi^2} \frac{\tilde{\lambda}(\triangle^\alpha)\tilde{T}(\triangle^\alpha)}{|\triangle^\alpha|}$$

At this point we can prove that $G(\alpha) > 1.01$ for all values $0.33 \leq \alpha \leq \pi/3$ by using Interval Arithmetic. There are many softwares and libraries which can be employed for this purpose. We selected *Octave*[1] (A free software that runs on GNU/Linux, macOS, BSD, and Windows) which provides a specific package called *Interval*.[2]

We covered the interval $\left[\frac{33}{100}, \frac{\pi}{3}\right]$ by a collection of 1001 intervals I_n with $n = 0, \ldots, 1000$, so that

$$I_n = \left[\frac{33}{100} + \frac{(n-1)}{1000}\left(\frac{\pi}{3} - \frac{33}{100}\right), \frac{33}{100} + \frac{(n+1)}{1000}\left(\frac{\pi}{3} - \frac{33}{100}\right)\right].$$

We choose the intersection of consecutive intervals to be non-empty. However, this is not necessary for Interval Arithmetic to be applicable: we could have chosen consecutive closed intervals with empty interior intersection. The current choice gives an extra, but unnecessary, check on rounding errors. Using the *Interval* package, we designed a code that for n going from 0 to 10^3 provides upper and lower bounds for $G(I_n)$ in terms of floating point numbers. This is performed in an automated way by standard and reliable algorithms. We established that the inequality $F(\alpha) > 1.01$ holds true on the whole interval $\left[\frac{33}{100}, \frac{\pi}{3}\right]$ by verifying it on I_n for all $n = 0, \ldots, 10^3$. □

For completeness we include the code below.

```
pkg load interval # load the package Interval
output_precision (6) # number of digits displayed
C=(9*pi/8)^(2./3)*2^(-1./3);

function K=G(x) # this is the definition of the function G(\alpha)
  Sum=0;
  for n = 0:10
    Sum = Sum + (2*n+1)^(-2)*(2*n+1+2*x/pi)^(-2)*(2*n+1-2*x/pi)^(-1);
  endfor
  K=(24./pi^2)*(1./16*(tan(x)-x)-8*x^4/pi^5*(Sum+1./(2.^7*10.^4)))*((cos(x/2))^2*(x/sin(x))*(pi/x+((9*pi/8)^(2./3)*2^(-1./3))*(pi/x)^(1/3))^2)/tan(x/2);
endfunction
control="OK"; # the variable control is set to "OK"
N=1000; # Number of intervals
Delta=(pi/3-0.33)/N; # 2Delta is the size of each interval
for n = 0:N
  n #print the value of n
  a=0.33+n*Delta;
  I=midrad(a,Delta) # I = interval with center in a and radius Delta
  J=G(I) # J is an interval which includes the image of I
  if(J>1.01) # check that (min J) > 1.01
  "so far inequality holds" # tell that everything is working fine
  elseif
  control="failure" # the variable control is set to "failure"
  break # in case of failure the cycle breaks
  endif
endfor
```

[1] John W. Eaton, David Bateman, Sren Hauberg, Rik Wehbring (2018). GNU Octave version 4.4.1 manual: A high-level interactive language for numerical computations. URL https://www.gnu.org/software/octave/doc/v4.4.1/

[2] Oliver Heimlich, GNU Octave Interval Package, https://octave.sourceforge.io/interval/, version 3.2.0, 2018-07-01. The interval package is a collection of functions for interval arithmetic. It is developed at Octave Forge, a sibling of the GNU Octave project.

Acknowledgments. The authors wish to thank Prof. Gerardo Toraldo for helpful discussions on Interval Arithmetic, and the referee for careful reading of the manuscript and helpful comments. The authors acknowledge support by the London Mathematical Society, Scheme 4 Grant 41636. MvdB was also supported by The Leverhulme Trust through Emeritus Fellowship EM-2018-011-9, and by an INDAM-GNAMPA Grant for visiting professors.

References

[1] M. van den Berg, Estimates for the torsion function and Sobolev constants, Potential Analysis **36** (2012), 607–616.

[2] M. van den Berg, Spectral bounds for the torsion function, Integral Equations and Operator Theory **88** (2017), 387–400.

[3] M. van den Berg, T. Carroll, Hardy inequality and L^p estimates for the torsion function, Bull. Lond. Math. Soc. **41** (2009), 980–986.

[4] M. van den Berg, V. Ferone, C. Nitsch, C. Trombetti, On Pólya's inequality for torsional rigidity and first Dirichlet eigenvalue, Integral Equations and Operator Theory **86** (2016), 579–600.

[5] A. Henrot, Extremum problems for eigenvalues of elliptic operators, Frontiers in Mathematics, Birkhäuser 2006.

[6] A. Henrot, I. Lucardesi, G. Philippin, On two functionals involving the maximum of the torsion function, ESAIM Control Optim. Calc. Var. **24** (2018), 1585–1604.

[7] H. G. Eggleston, Notes on Minkowski geometry (I): Relations between the circumradius, diameter, inradius and minimal width of a convex set, J. London Math. Soc. **33** (1958), 76–81.

[8] L. E. Payne, Bounds for solutions of a class of quasilinear elliptic boundary value problems in terms of the torsion function, Proc. Royal Soc. Edinburgh **88A** (1981), 251–265.

[9] G. Pólya, G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Ann. of Math. Stud. 27, Princeton University Press, Princeton, 1951.

[10] C. K. Qu, R. Wong, "Best possible" upper and lower bounds for the zeroes of the Bessel function $J_\nu(x)$, Trans. Am. Math. Soc. **351** (1999), 2833–2859.

[11] B. Siudeja, Isoperimetric inequalities for eigenvalues of triangles, Indiana Univ. Math. J. **59** (2010), 1097–1120.

[12] G. Talenti, Elliptic equations and rearrangements, Ann. Scuola Norm. Sup. Pisa **3** (1976), 697–718.

[13] S. P. Timoshenko, J. N. Goodier, *Theory of Elasticity*, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1951.

[14] H. Vogt, L_∞ estimates for the torsion function and L_∞ growth of semigroups satisfying Gaussian bounds, Potential Analysis, **51** (2019), 37–47.

[15] I. M. Yaglom, V. G. Boltyanskii, *Convex figures*, (Translated by P. J. Kelly, L. F. Walton), Holt, Rinehart and Winston, New York, 1961.